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HYDRODYNAMIC AND TRANSFER CHARACTERISTICS IN FREE INTERFACE FILM DUE TO TIME-DEPENDENT DISTURBANCE AT THE ENTRY

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Abstract Theoretical analysis of the hydrodynamic and transport characteristics at the entry region of free surface films is presented. The governing equations for mass continuity, liquid motion and heat transfer across the film are newly formulated in terms of the (usually measurable) parameters at the interface, by applying the latter as a so-called collocation line. Inertia terms are retained as required in problems associated with rapid disturbances. A time-dependent disturbance is applied at the entry and the effects of its propagation on the film- and transfer-characteristics down-stream are evaluated and discussed.

The collocation forms of the governing equations might be of an interest for further film analyses.

NOMENCLATURE

ai, bi,	constants;
d,	distance from free interface [m];
f,	rate parameter (or frequency of disturbance at $x = 0$) [1/s];
F_0,	Fourier number;
g_x,	gravitation component in x direction [m/s^2];
h,	film thickness [m];
H,	normalized film thickness;
H_f,	final film thickness;
h_N,	Nusselt film thickness [m];
k,	heat conductivity [W/m^2(K/m);
l,	liquid heat of vaporization [δ/kg];
p,	pressure ($p' = p - \rho g_x x$) [N/m^2];
Q,	local volumetric flow rate;
q_w,	wall heat flux [W/m^2];
Q_w,	normalized wall-heat flux;
Q_A,	average normalized wall-heat flux with respect to distance;
Q_{AA},	average normalized wall-heat flux with respect to distance and time;
Re,	Reynolds number;
s,	film surface velocity [m/s];
S,	normalized film surface velocity;
t,	time [2];
T,	temperature [K];
T_w,	wall temperature [K];
T_v,	film free interface temperature [K];
u,	velocity in the x direction [m/s];
v,	velocity in the y direction [m/s];
We,	Weber number;
X,	normalized distance in the x direction;
X_c,	maximum distance that the disturbance advanced at time τ_F.

Greek symbols

α,	thermal diffusivity [m^2/s];
γ,	local mass flow rate [kg/sm];
γ_i,	inlet mass flow rate [kg/sm];
Γ,	normalized mass flow rate;
θ_w,	temperature drop ($T - T_v$) [K];
Θ_w,	normalized θ_w;
λ,	Pohlausen's parameter;
μ,	viscosity [NS/m^2];
ν,	kinematic viscosity [m^2/s];
ρ,	density [kg/m^3];
σ,	surface tension [N/m];
τ,	normalized time;
τ_F,	normalized time for reaching H_f at $X = 0$;
$\bar{\bar{\tau}}$,	stress tensor [N/m^2];
ψ,	stream function [m^2/s].

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1. INTRODUCTION

FLUID motion and transport in thin free-surface liquid films are of fundamental interest in much basic industrial equipment, e.g. in steam condensers, wetted wall columns, liquid-film evaporators and other processes involving interfacial heat and mass transfer. Of particular interest are the horizontal type evaporator-condenser systems [1, 2], where either the liquid condensate or the evaporating film flows between vertically neighbouring tubes by means of falling droplets, which detach at the bottom of one tube and 'rain' on the top of the tube below. Thin films are thus sustained by successive drops and at the time-interval between two consecutive drops, the liquid drains downward [3].

Due to this kind of constantly undeveloped flow, the film characteristics and the transfer rates are complex periodical functions of time. Obviously, the film characteristics (thickness of that layer which remains on the surface and its drainage velocity) as well as the transport characteristics are influenced most at the under-developed region which follows the initial boundary where the dripping, and hence the change of feeding rate, occurs. The contribution of this under-developed entry region is of practical importance in cases where the transfer areas are relatively short. For instance, the film-drainage path on a horizontal-tube bundle comprises consecutive sections each being half of the conduit periphery.

Heat, mass and momentum transport for laminar confined flows in the entrance region of closed conduits has received comprehensive attention [4–9]. However, the entrance region for laminar free-surface flows has received relatively little attention. The entrance effects on mass-transfer processes in laminar free-surface falling liquid films have been accounted for by Scriven [10] and Toor [11]. Laminar flow along a vertical wall has been analysed by Hassan [12], utilizing the integral equation of momentum and has been improved later on by Haugen [13] assuming a developing boundary layer at the entrance. Direct integration of the governing differential equation (by using finite difference methods) has been presented by Bruley [14]. Fyrther numerical solutions of the equation of motion in the entrance region have been reported by Cerro and Whitaker [15]. In all the above studies, the effect of viscous drag at the interface has been neglected. This has been accounted for later by Murty and Saster [16].

The present study represents a theoretical analysis of the unsteady hydrodynamic and transport characteristics in thin films. Particular emphasis is made on the effect of a time-dependent disturbance applied at the entry region and its propagation downstream. For instance, reduction of film thickness, velocity or feed flow rate at the entry in a step-change mode or as a function of time.

The equations of continuity, motion and energy for free-interface thin films are first formulated in terms of the free liquid-interface parameters by applying the latter as a collocation line. The collocation equations obtained are simultaneously solved for the instantaneous film thickness, velocity at its interface and the associated transfer rates, using Lax–Wendroff numerical schemes.

2. THE PHYSICAL MODEL

A schematic description of the physical model and coordinates are illustrated in Fig. 1. An initially plane laminar film flows down a vertical wall emerging at $x = 0$. The feed rate at $x = 0$ is uniformly distributed in the z direction at the times $t = 0$ and it rates at γ_i per unit width of the wall. However, for $t > 0$ the film-feed flow rate is changed in either a stepwise or time-dependent manner. Note that a zero gas–liquid shear is assumed at the film free interface, denoted here by $h(x, t)$.

The thermal conditions are defined by either a constant wall temperature, T_w, at the surface $y = 0$, or a constant heat flux, q_w normal to this place. The liquid feed is distributed at an inlet temperature, T_v and it is assumed that the temperature at the film free interface

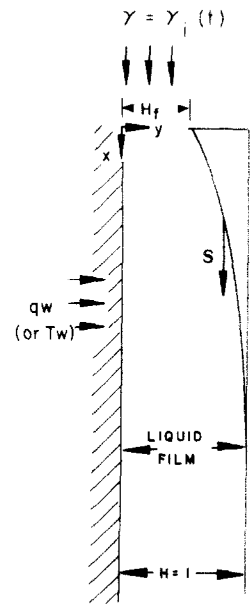

FIG. 1. Schematic description of physical model.

is maintained constantly at T_v (corresponding to a constant saturation pressure p_v).

2.1. *The conditions at the gas–liquid interface*

With reference to Fig. 1 the geometry of the gas–liquid interface is given by the normal and tangential unit vectors:

$$\bar{n} = (-h_x, 1)/|\nabla d|, \tag{1a}$$

$$\bar{t} = (1, h_x)/|\nabla d|, \tag{1b}$$

where subscript x denotes derivation in the x direction and d denotes the distance from the free interface given by:

$$d(x, y, t) = y - h(x, t); \quad |\nabla d| = (1 + h_x^2)^{1/2}. \tag{2}$$

Assuming now two dimensional flow (uniformity in the z direction) and utilizing boundary-layer approximations the film stress tensor is:

$$\bar{\bar{\tau}} = \begin{vmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{vmatrix} = \begin{vmatrix} -p + 2\mu\dfrac{\partial^2\psi}{\partial x \partial y} & \mu\dfrac{\partial^2\psi}{\partial y^2} \\ \mu\dfrac{\partial^2\psi}{\partial y^2} & -p - 2\mu\dfrac{\partial^2\psi}{\partial x \partial y} \end{vmatrix} \tag{3}$$

where p is the pressure and $\psi(x, y, t)$ is a stream function (defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$). Utilizing equations (1a) and (1b), the stress vector at the gas–liquid interface is:

$$\bar{f} = \bar{\bar{\tau}} \cdot \bar{n} = \frac{1}{[1 + h_x^2]^{1/2}} \left| Ph_x - 2\mu h_x \frac{\partial^2\psi}{\partial x \partial y} + \mu\frac{\partial^2\psi}{\partial y^2}, \quad -p - \mu h_x \frac{\partial^2\psi}{\partial y^2} - 2\mu\frac{\partial^2\psi}{\partial x \partial y} \right| \tag{4}$$

whereby the normal and tangential component to the interface are:

$$\bar{f}_n = (\bar{\bar{\tau}} \cdot \bar{n}) \cdot \bar{n} = -p + \frac{2^{\mu}}{1+h_x^2}$$

$$\left[-\frac{\partial^2\psi}{\partial x \partial y} - h_x \frac{\partial^2 \psi}{\partial y^2} + h_x^2 \frac{\partial^2\psi}{\partial x\partial y}\right]\Bigg|_{y=h}, \quad (5a)$$

$$\bar{f}_t = (\bar{\bar{\tau}} \cdot \bar{n}) \cdot \bar{t} = \mu \frac{1-h_x^2}{1+h_x^2} \frac{\partial^2\psi}{\partial y^2}\Bigg|_{y=h}$$

$$- \frac{4\mu h_x}{1+h_x^2} \frac{\partial^2\psi}{\partial x\partial y}\Bigg|_{y=h}. \quad (5b)$$

Following the boundary-layer approximation, all terms of $\partial^2\psi/\partial x\partial y$ may be neglected and equations (5) reduce to

$$\bar{f}_n = -p - \frac{2^{\mu} hx}{1+h_x^2} \frac{\partial^2\psi}{\partial y^2}\Bigg|_{y=h}, \quad (6a)$$

$$\bar{f}_t = \mu \frac{1-h_x^2}{1+h_x^2} \frac{\partial^2\psi}{\partial y^2}\Bigg|_{y=h}. \quad (6b)$$

2.2. *The governing equations of motion*

The simplified Navier–Stokes equations (known as Prandtl's boundary-layer equations) for two-dimensional non-steady flow in the $x - y$ plane are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + g_x, \quad (7)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (8)$$

As it is often convenient, the stream function is introduced, so that the continuity equation (8), is satisfied automatically, and in addtion, the equation of momentum is transformed into a single-unknown third order partial-differential equation.

$$\frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\frac{\partial^3\psi}{\partial y^3};$$

$$p' = p - \rho g_x x. \quad (9)$$

Differentiation of equation (9) with respect to y results in the vorticity equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial y^2}\right) + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial y^2}\right)$$

$$- \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2\psi}{\partial y^2}\right) = \nu\frac{\partial^4\psi}{\partial y^4}. \quad (10)$$

The prescribed stream function ψ (x, y, t) is expanded now at (x, t) into six powers of y, with coefficients a_i (x, t) which are functions of x and t, the expansion being:

$$\psi(x,y,t) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4^4 y^4 + a_5 y^5.$$

Some of the coefficients can be selected in accordance with the appropriate boundary conditions, whereas the others are connected through conditions of compatibility at either the wall or the liquid free surface. In terms of the stream function, the boundary and compatibility conditions are:

$$y=0; \quad \frac{\partial\psi}{\partial y} = 0 \qquad a_1 = 0, \quad (11a)$$

$$y=0; \quad \psi = 0 \qquad a_0 = 0, \quad (11b)$$

$$y=0; \quad \frac{1}{\rho}\frac{\mathrm{d}p'}{\mathrm{d}x} = \nu\frac{\partial^3\psi}{\partial y^3} \qquad a_3 = \frac{1}{6\mu}\frac{\mathrm{d}p'}{\mathrm{d}x} = -\lambda, \quad (11c)$$

$$y=h; \quad \frac{\partial^4\psi}{\partial y^4} = 0 \qquad a_4 + 5a_5 h = 0, \quad (11d)$$

$$y=h; \quad \frac{\partial^2\psi}{\partial y^2} = 0$$

$$a_2 - 3\lambda h + 6a_4 h^2 + 10 a_5 h^3 = 0, \quad (11e)$$

$$y=h; \quad \frac{\partial\psi}{\partial y} = s$$

$$2a_2 h - 3\lambda h^2 + 4a_4 h^3 + 5a_5 h^4 = s, \quad (11f)$$

$$y=h; \quad P_v = \sigma\frac{\partial^2 h/\partial x^2}{[1+(\partial h/\partial x)^2]^{3/2}} - \bar{f}_n;$$

$$P_v = \text{const.} \quad (11g)$$

The first two conditions at the wall are due to the absence of fluid slip and an impermeable solid surface. These two combined with the momentum equation (9) result in a third relationship at the wall. Equation (11e) is a compatibility condition which results from equation (6b) for the case of no tangential shear stress at the free interface. The fourth relationship is again a compatibility condition which expresses the complete time derivative of the vorticity, moving with a fluid particle along the free-interface streamline. This expression follows from condition (11e).

Finally, equation (11f) implies that a new parameter s is introduced, in terms of which all other coefficients are related. Thus, the solutions of equations (11) yield:

$$\begin{aligned} a_2 &= (4s + 3\lambda h^2)/5h, \\ a_4 &= -(s - 3\lambda h^2)/5h^3, \\ a_5 &= (s - 3\lambda h^2)/25h^4, \end{aligned} \quad (12)$$

where λ, Pohlhausen's parameter, is given by equation (11c). However, since the external pressure P_v is constant, the pressure gradient $\mathrm{d}P/\mathrm{d}x$ across the film is equal $\mathrm{d}\bar{f}_n/\mathrm{d}x$. The latter is evaluated by differentiating equation (11g).

The equation of motion (9), applied at the line $y = h$ (so-called collocation line), now reads:

$$\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = \frac{6}{5}g - \frac{12}{5}\nu\frac{s}{h^2} + \frac{6}{5}\frac{\sigma}{\rho}\frac{\partial}{\partial x}\left[\frac{h_{xx}}{(1+h_x^2)^{3/2}}\right]. \quad (13)$$

Note that s is the local velocity at the film interface in the direction of the main flow.

Also, the local mass flow rate can be obtained in terms of s by integrating the velocity profile over the film thickness, thus:

$$\gamma = \rho Q = \rho \int_0^h \frac{\partial \psi}{\partial y} \mathrm{d}y = \rho \left\{ \frac{16}{25} sh + \frac{gh^3}{75\nu} + \frac{\sigma h^3}{75\mu} \frac{\partial}{\partial x} \left[\frac{h_{xx}}{(1+h_x^2)^{3/2}} \right] \right\}. \quad (14)$$

where Q is the local volumetric flow rate.

2.3. *The energy equation*

Substituting the stream function components, equations (3) into the two-dimensional energy equations and differentiating with respect to y yields:

$$\frac{\partial^2 T}{\partial t \partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial T}{\partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial T}{\partial y} + \frac{\partial \psi}{\partial y} \frac{\partial^2 T}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 T}{\partial y^2} = \alpha \frac{\partial^3 T}{\partial y^3}. \quad (15)$$

Here too, T is expanded in (five) power series

$$\sum_{i=0}^{5} b_i y^i$$

and the various coefficients are determined, based on boundary and compatibility conditions. These are:

$$y = 0, \qquad T = T_w, \quad (16a)$$

$$y = 0, \qquad \frac{\partial T}{\partial y} = -q_w/k, \quad (16b)$$

$$y = 0, \quad \alpha \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial t}, \quad (16c)$$

$$y = 0, \quad \alpha \frac{\partial^3 T}{\partial y^3} = \frac{\partial^2 T}{\partial t \partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial T}{\partial x}, \quad (16d)$$

$$y = h, \qquad T = T_v, \quad (16e)$$

$$y = h, \qquad \frac{\partial^2 T}{\partial y^2} = 0. \quad (16f)$$

Note that conditions (16c) and (16d) are obtained from the energy equation and its differentiated form, equation (15) at the wall. The last condition results since the free interface is a stream line whereupon the complete time derivation is zero. Applying the relationships (16a)–(16d) and (16f) on the temperature power series yields the coefficients in terms of the temperature and thermal flux at the wall:

$$b_0 = \theta_w; \quad b_1 = -q_w/k; \quad b_2 = \frac{1}{2\alpha} \frac{\partial \theta}{\partial t};$$

$$b_3 = -\frac{5}{6\alpha h} \frac{\partial \theta_w}{\partial t} - \frac{2\theta_w}{h^3} + \frac{2q_w}{kh^2}; \quad (17)$$

$$b_4 = \frac{1}{3\alpha h^2} \frac{\partial \theta_w}{\partial t} + \frac{\theta_w}{h^4} - \frac{q_w}{kh^3},$$

where $\theta_w = T_w - T_v$.

The remaining relationship, equation (16d) is now treated whereby the LHS is evaluated utilizing the power series of T (or θ) with equation (17) whereas its RHS is evaluated from the power series of ψ with equation (12). This results in a θ_w (or q_w) equation in terms of the film thickness and the velocity at the free interface:

$$\frac{\partial \theta_w}{\partial t} + \left(\frac{8}{25} s + \frac{g}{25\nu} h^2 \right) \frac{\partial \theta_w}{\partial x} = \frac{12\alpha}{5kh} \left(q_w - \frac{k\theta_w}{h} \right) + \frac{h}{5k} \frac{\partial q_w}{\partial t} - \frac{\sigma h^2}{25\mu} \frac{\partial}{\partial x} \left[\frac{h_{xx}}{(1 + h_x^2)^{3/2}} \right] \frac{\partial \theta_w}{\partial x} \quad (18)$$

If a constant temperature, T_w is maintained at the solid-film interface, equation (12) reduces to:

$$\frac{\partial q_w}{\partial t} = \frac{12\alpha}{h^2} \left(q_w - \frac{k\theta_w}{h} \right); \quad \theta_w = \text{const.} \quad (19)$$

However, for the case of a constant heat flux, q_w, at the wall, equation (18) reduces to:

$$\frac{\partial \theta_w}{\partial t} + \left(\frac{8}{25} s + \frac{gh^2}{25\nu} \right) \frac{\partial \theta_w}{\partial x} = \frac{12\alpha}{5kh} \left(q_w - \frac{k\theta_w}{h} \right) + \frac{\sigma h^2}{25\mu} \frac{\partial}{\partial x} \left[\frac{h_{xx}}{(1 + h_x^2)^{3/2}} \right] \frac{\partial \theta_w}{\partial x}. \quad (20)$$

The film-free interface is described by $[x, h(x, t)]$ and the time derivatives of these coordinates $[0, \partial h/\partial t]$. Similarly, the interface motion is described by a vector $[s, -\partial \psi/\partial x]$, while the normal component to the film interface, $\bar{n}$, is given by equations (1) and (2). Moreover, if there exists a heat flux, q_h, at the free interface, evaporation occurs at an amount of $q_h/\rho l$, where l is the liquid heat of vaporization. Therefore, the normal interface motion due to evaporation $q_h \bar{n}/\rho l$ must be subtraced from the interface motion in the normal direction. The continuity condition at the free interface is, thus:

$$\left[0, \frac{\partial h}{\partial t} \right] \bar{n} = [s, -\partial \psi/\partial x]_h \bar{n} - \left(\frac{q}{\rho l} \right)_h \bar{n}. \quad (21)$$

The flux vector at the interface is obtained from the temperature-gradient vector at this line. Utilizing the temperature constants, equation (17) in $q_h = -k(\partial T/\partial y)_{y=h}$.

$$\left(\frac{q_h}{\rho l} \right) \bar{n} = \frac{-1}{\rho l} \left[q_w - \frac{2k\theta_w}{h} - \frac{kh}{6\alpha} \frac{\partial \theta_w}{\partial t} \right] \Bigg/ \left[1 + \left(\frac{\partial h}{\partial x} \right)^2 \right]^{1/2}. \quad (22)$$

Also from equations (11) and (12) one obtains:

$$\left(\frac{\partial \psi}{\partial x} \right)_{y=h} = \frac{16}{25} h \frac{\partial s}{\partial x} - \frac{g}{25} s \frac{\partial h}{\partial x} + \frac{6}{25} \lambda h^2 \frac{\partial h}{\partial x} + \frac{2}{25} h^3 \frac{\partial \lambda}{\partial x}. \quad (23)$$

Combining equations (22) and (23) into equation (21), the latter becomes:

$$\frac{\partial h}{\partial t} + \left(\frac{16}{25}s + \frac{g}{25\nu}h^2\right)\frac{\partial h}{\partial x}$$
$$= -\frac{16}{25}h\frac{\partial s}{\partial x} + \frac{q_w}{\rho l} - \frac{2k\theta_w}{\rho l h} - \frac{kh}{\alpha \rho l}\frac{\partial \theta_w}{\partial t}$$
$$- \frac{\sigma}{25\mu}h^2\frac{\partial}{\partial x}\left[\frac{h_{xx}}{(1+h_x^2)^{3/2}}\right]\frac{\partial h}{\partial x}$$
$$- \frac{\sigma}{75\mu}h^3\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 h/\partial x^2}{[1+(\partial h/\partial x)^2]^{3/2}}\right). \tag{24}$$

It is worth noting that equation (24) can easily be switched to an equivalent integral continuity form by utilizing equation (14). Differentiating the latter and substituting in equation (24) yields

$$\frac{\partial h}{\partial t} + \frac{1}{\rho}\frac{\partial \gamma}{\partial x} = -\left(q_w - \frac{2k\theta_w}{h} - \frac{kh}{\alpha}\frac{\partial \theta_w}{\partial t}\right)\Big/\rho l. \tag{25}$$

Obviously, if the heat-transfer resistance lies in a sublayer thinner than the liquid film no evaporation will occur, i.e. $q_N\bar{n}/\rho l$ and equation (25) reduces to:

$$\frac{\partial h}{\partial t} + \frac{1}{\rho}\frac{\partial \gamma}{\partial x} = 0, \tag{26}$$

which is the usual continuity equation.

The equation of motion, equation (13) with the continuity equation (25) or (26) determine the film hydrodynamic behaviour. When thermal-boundary conditions are applied, these must be solved with temperature equation (18). The latter may be replaced by either equation (19) or (20) for cases of constant temperature and constant heat flux at the wall, respectively.

2.4. *The normalized governing equations*

The governing equations are now normalized utilizing the following non-dimensional variables:

$$S = s/\bar{U}_N; \quad H = h/h_N; \quad \Gamma = \gamma/(\rho \bar{U}_N h_N)$$
$$X = x/L; \quad \tau = t/(L/\bar{U}_N); \quad L = h_N(We)^{1/2}$$
$$Re = \frac{\bar{U}_N h_N}{\nu}; \quad We = \frac{9\sigma\nu^2}{\rho g^2 h_N^5}; \quad \bar{U}_N = \frac{\gamma_i}{h_N} = \frac{g h_N^2}{3\nu}$$
$$Q_N = q_w/q_N; \quad \Theta_w = \theta_w/(q_N h_N/k); \quad F_0 \frac{\alpha(L/\bar{U}_N)}{h_N^2}, \tag{27}$$

where Re, W_e and F_0 are the Reynolds, Weber and Fourier numbers, respectively, and h_N, $\bar{U}_N$ are the film thickness and the average velocity, respectively, of a laminar smooth Nusselt film, corresponding to initial feed-flow rate, γ_i. Similarly, q_N is the heat flux which would result for constant wall temperature, $\theta_w = T_w - T_v$, under the Nusselt assumption of a linear temperature profile across the film of thickness h_N. Thus $q_N = k(T_w - T_v)h_N$, is used in normalizing equation (19). On the other hand, for constant heat flux q_w at the wall, equation (20) is normalised by the temperature gradient, Θ_N which would result across a Nusselt film, h_N by applying q_w (i.e. $\Theta_N = q_w h_N/k$).

Note also that the characteristic length L is a wave length in the neighbourhood of the most amplified wave. The preference of normalizing x by L rather than h_N is explained below. Equations (13), (19), (20) and (25) in non-dimensionalized form are, respectively:

$$\frac{\partial S}{\partial \tau} + S\frac{\partial S}{\partial X} = \frac{6}{5}\frac{(We)^{1/2}}{Re}\left(3 - 2\frac{S}{H^2}\right)$$
$$+ \frac{6}{5}\frac{\partial}{\partial X}\left\{\frac{\partial^2 H/\partial X^2}{[1+(\partial H/\partial X)^2/We]^{3/2}}\right\}, \tag{13'}$$

$$\frac{\partial Q_w}{\partial \tau} = -\frac{12}{H^2}F_0\left(Q_w - \frac{1}{H}\right), \tag{19'}$$

$$\frac{\partial \Theta_w}{\partial t} + \left(\frac{8}{25}S + \frac{3}{25}H^2\right)\frac{\partial \Theta_w}{\partial X} = \frac{12F_0}{H}\left(1 - \frac{\Theta_w}{H}\right)$$
$$- \frac{1}{25}\frac{Re}{(We)^{1/2}}H^2\frac{\partial}{\partial X}\left\{\frac{\partial^2 H/\partial X^2}{[1+(\partial H/\partial X)^2/We]^{3/2}}\right]\frac{\partial \Theta_w}{\partial X}, \tag{20'}$$

$$\frac{\partial H}{\partial \tau} + \frac{\partial \Gamma}{\partial X} = 0. \tag{25'}$$

The dimensionless local flow rate is, by equation (14):

$$\Gamma = \frac{16}{25}SH + \frac{1}{25}H^3 + \frac{1}{75}\frac{Re}{(We)^{1/2}}H^3\frac{\partial}{\partial X}$$
$$\times \left\{\frac{\partial^2 H/\partial X^2}{[1+(\partial H/\partial X)^2/We]^{3/2}}\right\}. \tag{14'}$$

3. METHOD OF SOLUTION

To solve equations (13′), (19′), (20′) and (25′) we first neglect the surface-tension effect (terms including the third derivative of H with respect to X are dropped). Substituting equation (14′) in (25′), the following set of equations is obtained.

$$\frac{\partial S}{\partial \tau} + S\frac{\partial S}{\partial X} = \frac{6}{5}\frac{(We)^{1/2}}{Re}\left(3 - 2\frac{S}{H^2}\right), \tag{13*}$$

$$\frac{\partial H}{\partial \tau} + \frac{16}{25}S\frac{\partial H}{\partial X} + \frac{16}{25}H\frac{\partial S}{\partial X} + \frac{3}{25}H^2\frac{\partial H}{\partial X} = 0, \tag{25*}$$

$$\frac{\partial Q_w}{\partial \tau} = -\frac{12}{H^2}F_0\left(Q_w - \frac{1}{H}\right), \tag{19*}$$

$$\frac{\partial \Theta_w}{\partial \tau} + \left(\frac{8}{25}S + \frac{3}{25}H^2\right)\frac{\partial \Theta_w}{\partial X}$$
$$= \frac{12}{5}\frac{F_0}{H}\left(1 - \frac{\Theta_w}{H}\right). \tag{20*}$$

The first two equations are first solved simultaneously; the other two can be solved using the solution of S and H.

The first two equations are a set of non-linear hyperbolic partial-differential equations having the form

$$\frac{\partial}{\partial \tau}\begin{pmatrix} S \\ H \end{pmatrix} + \begin{pmatrix} S & 0 \\ \frac{16}{25}H & \frac{16}{25}S + \frac{3}{25}H^2 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} S \\ H \end{pmatrix} = \begin{pmatrix} 1.2\frac{(We)^{1/2}}{Re}\left(3 - \frac{2S}{H^2}\right) \\ 0 \end{pmatrix}$$

This system is hyperbolic system with real eigenvalues

$$\left[S, \frac{16}{25}S + \frac{3}{25}H^2\right].$$

The initial conditions used are:

$$\tau = 0, \quad X > 0; \quad H = 1.0, \quad S = 1.5, \tag{28a}$$

$$\tau < \frac{1}{ft_r}, \quad X = 0; \quad H = 1 - (1 - H_f)ft_r\tau, \quad S = 1.5H^2, \tag{28b}$$

$$\tau > \frac{1}{ft_r}, \quad X = 0; \quad H = H_f, \tag{28c}$$

where H_f is the final thickness, t_r is a reference time constant, f is a rate parameter which determines the rate of change of the feed at $X = 0$.†

No stable numerical solutions have been found yet for this system in its present form, but some attempts are still underway. A solution for a special case has been obtained by dropping the non-homogeneous term

$$1.2.\frac{(We)^{1/2}}{Re}\left(3 - 2\frac{S}{H^2}\right)$$

which forms nonlinear coupling between the two equations. The neglect of this term is reasonable for large values of Re. Also for large f the final values of τ and X are small [equation (28b)] and hence the contribution of the non-homogeneous term in the integration of equation (13th) is small. The solution is obtained by first bringing the system to a conservation form as follows.

$$\frac{\partial S}{\partial \tau} + \frac{\partial}{\partial X}\left(\frac{S^2}{2}\right) = 0, \tag{29a}$$

$$\frac{\partial H}{\partial \tau} + \frac{\partial}{\partial X}(0.64SH + 0.04H^3) = 0. \tag{29b}$$

This system is solved numerically using the Lax–Wendroff method which is of second order accuracy. Intermediate values of the functions (S, H) are calculated initially. Then the value of S, H, on the next grid point are calculated.

If n is the time index, i is the distance index such that $t = n\Delta t$ and $X = i\Delta x$ where Δt and Δx are the temporal and spatial increments respectively,

$$S_{i+1/2}^{n+1/2} = 0.5(S_i^n + S_{i+1}^n) - 0.5\left[\left(\frac{S^2}{2}\right)_{i+1}^n - \left(\frac{S^2}{2}\right)_i^n\right]\frac{\Delta\tau}{\Delta x}, \tag{30a}$$

$$S_{i-1/2}^{n+1/2} = 0.5(S_i^n + S_{i-1}^n) - 0.5\left[\left(\frac{S^2}{2}\right)_i^n - \left(\frac{S^2}{2}\right)_{i-1}^n\right]\frac{\Delta\tau}{\Delta x} \tag{30b}$$

and

$$S_i^{n+1} = 0.5\,S_i^n - \left[\left(\frac{S^2}{2}\right)_{i+1/2}^{n+1/2} - \left(\frac{S^2}{2}\right)_{i-1/2}^{n+1/2}\right]\frac{\Delta\tau}{\Delta x}. \tag{30c}$$

Similar expressions for H are obtained from equation (29b).

Simultaneously, other variables and functions are calculated, including: wall temperature, Θ_w; wall heat flux, Q_w; mass flow rate, Γ; average heat flux with distance, Q_A; average heat flux with distance and time, Q_{AA}; etc.

The temperature at the wall, Θ_w is calculated from equation (20*) using the Lax method and forward derivative with distance for increasing stability.

$$Q_i^{n+1} = 0.5(Q_{i+1}^n + Q_{i-1}^n) + \Delta\tau(A_T - B_T) \tag{31}$$

where

$$A_T = 2.4\,F_0\left(1 - \frac{\Theta_{wi}}{H_i^n}\right)$$

and

$$B_T = 0.04(8\,S_i^n + 3H_i^{n2})(\Theta_{i+1}^n - \Theta_i^n)/\Delta x.$$

The heat flux at the wall, Q_w is calculated from equation (19*) using a simple central derivative

$$\frac{Q_{w_i}^{n+1} - Q_{w_i}^n}{\Delta\tau} = \frac{-12F_0}{H^2}\left(Q_w - \frac{1}{H}\right) \tag{32}$$

where, for $H = 1/2\,(H_i^n + H_{i+1}^n)$ and $Q_w = 1/2\,(Q_{wi}^n + Q_{w_{i+1}}^n)$ yields

$$Q_{w_i}^{n+1} = Q_{w_i}^n\left[1 - 6F_0\frac{\Delta\tau}{H^2}\right) + 12F_0\frac{\Delta\tau}{H^2}\Big] \Big/ (1 + 6F_0\Delta\tau/H^2). \tag{33}$$

The heat-transfer coefficient U in the case of constant heat flux is proportional to $1/T_w$. In their dimensionless values both U and T_w are initially 1, thus

$$U = 1/T_w. \tag{34}$$

RESULTS AND DISCUSSION

The following parameters have been studied:

1. Re_N number has been varied between 100 and 600.
2. Final film thickness is chosen at either 0.5 or 0.05 of its initial value, h_N; thus, $H_f = 0.5$ or 0.05.
3. Frequency factor was varied between $f = 2.5$ and 20.

† f corresponds e.g. to the drop frequency at $x = 0$ and $1/f$ is the drainage time between successive drops.

A typical output of the calculated parameters is demonstrated in Fig. 2 for intermediate values of Reynold's number and rate parameter, $Re_N = 400$ and $f = 10$. During the time interval, $1/f$, the film thickness at the interval point is reduced [according to equation (28b)] to half of its initial value. Shortly after the change at $X = 0$ occurs, a relatively steep front, ending with $H = 1.0$ at $X = X_c$, is set in motion in the main direction of flow. This liquid front advances with speed of approximately $1.5(H^2)$, which is the free surface velocity corresponding to Nusselt analysis. However, the local free liquid surface (Fig. 2b), for the drainage region, $0 < X < X_c$, strongly increases with the distance downstream (even for this part where the film thickness is constant), finally approaching its initial value of 1.5 at $X \to X_c$. Consequently, the local film flow rate, Q, varies with X in a somewhat milder manner (Fig. 2c).

The local wall temperature (obtained with constant wall flux) and the local flux at the wall (obtained with constant wall temperature) are illustrated in Figs. 2d and 2e, respectively. For a constant flux at the wall, a decrease in the film thickness yields a decrease in the local temperature at the wall. On the other hand, for a

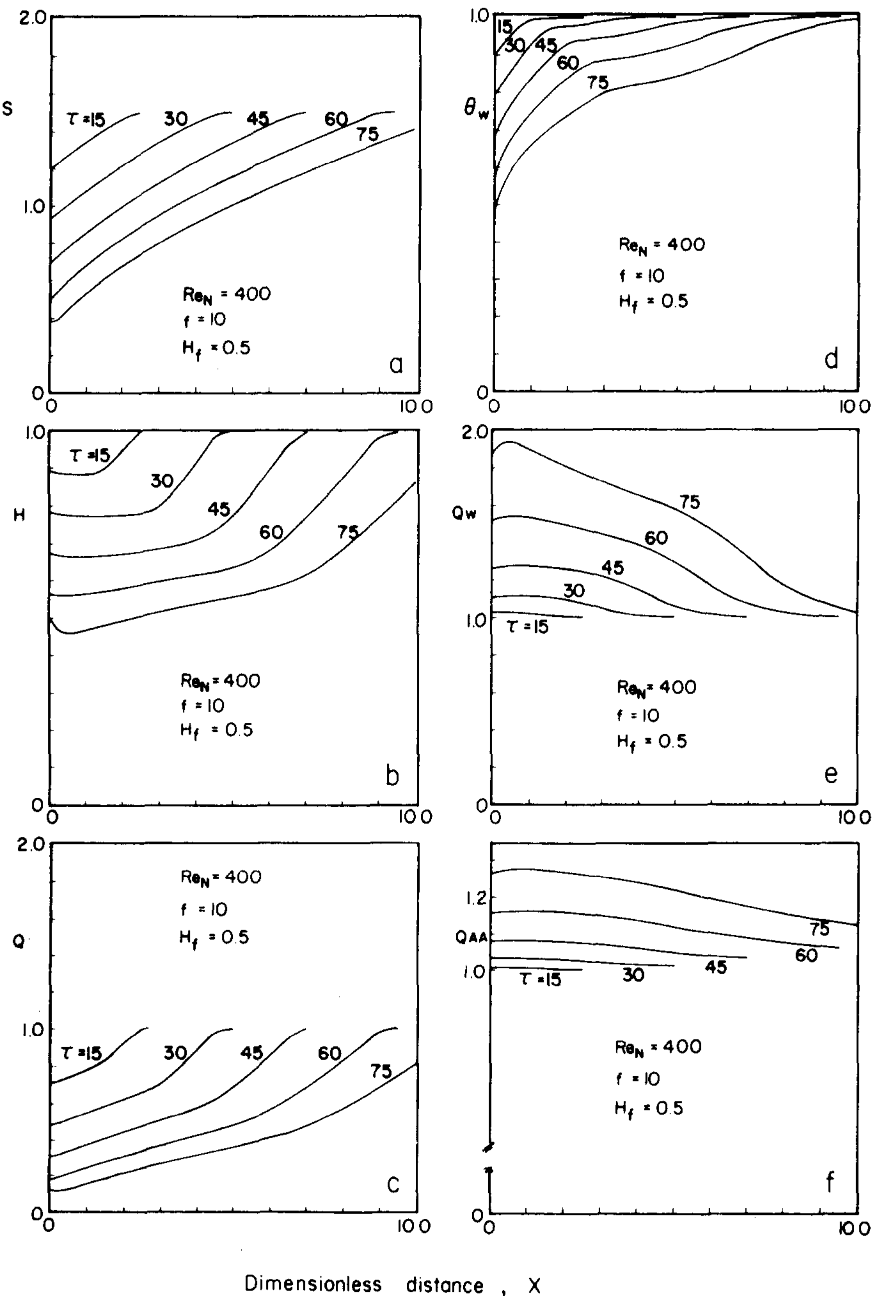

FIG. 2. Hydrodynamic and transport characteristics for relatively slow disturbance at $X = 0$ ($H_F = 0.5$).

constant wall temperature, the reduction in the film thickness brings about an enhanced heat flux at the wall. It is thus of a great interest to evaluate the total enhancement, Q_{AA} obtained over a distance X at a certain time, τ. As is shown in Fig. 2f, at time $1/f$ (when the film at $X = 0$ reaches half of its initial value) the total enhancement in the heat flux is between 25 per cent at $X = 0$ and 15 per cent at $X = X_c$. For a shorter plate of $L < X_c$ the expected enhancement is higher than 15 per cent.

The results presented in Fig. 2 correspond to a relatively slow variation of the film thickness at $X = 0$, whereby it is reduced to half of its initial thickness at time of $1/f$ $(= 1/10)$. A similar presentation is given in

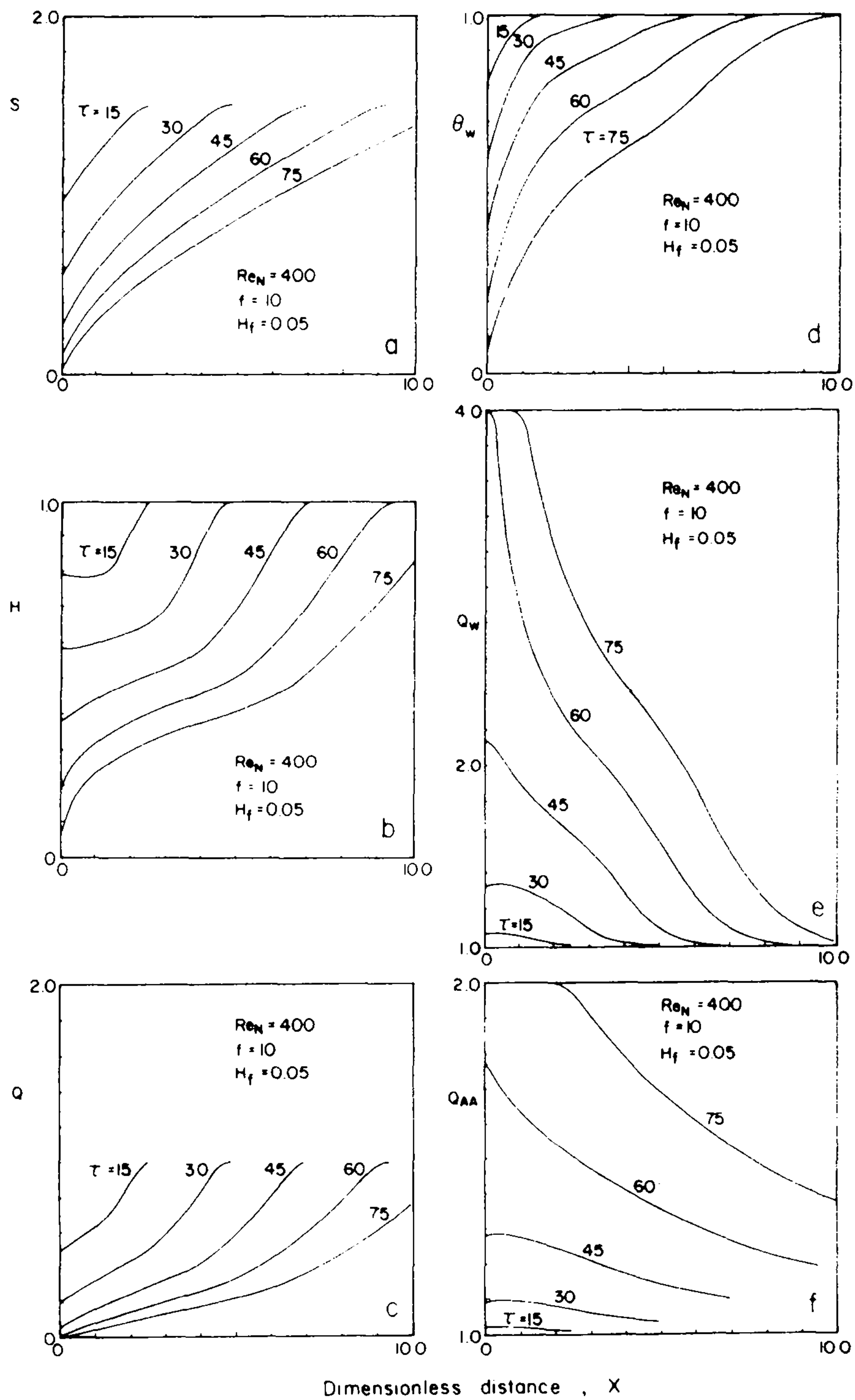

FIG. 3. Hydrodynamic and transport characteristics for relatively fast disturbance at $X = 0$ $(H_F = 0.05)$.

Fig. 3 for the case of rapidly-diminishing liquid film at $X = 0$, approaching 5 per cent of its initial value at the same period of time ($1/f = 1/10$). It is interesting to note that due to the relatively thin film associated here the effect of the solid wall is to reduce the first liquid front near $X = X_c$, while a new one appears at almost $X \rightarrow 0$. This effect can be seen also in Fig. 2a where the steep front obtained closed to $X = 0$ slopes more and more with time. As is to be expected, the large variations in the local film thickness and in the velocity accordingly, affect the temperature or the heat flux at the wall. For instance, at time τ_F, the heat flux at the wall (for constant wall temperature) may be enhanced by some 40 per cent for a plate of length $L = X_c$ and are even doubled for shorter plates.

The various physical variables, S, H, Θ_w, Q_w, Q_A and Q_{AA} are compared in Figs. 4–9 for a wide range of each of the basic parameters Re_N, $1/f$ and H_f. Note that Q_A represent instantaneous enhancement in the transfer rate averaged over the distance X. Since the film thickness decreases with time, this enhancement is more and more pronounced as the disturbance at $X = 0$ propagates downstream. For completeness, the averaging of Q_A over the time elapsed is shown in Fig. 9. Thus, Q_{AA} represents the overall enhancement at time τ_F and over a length X.

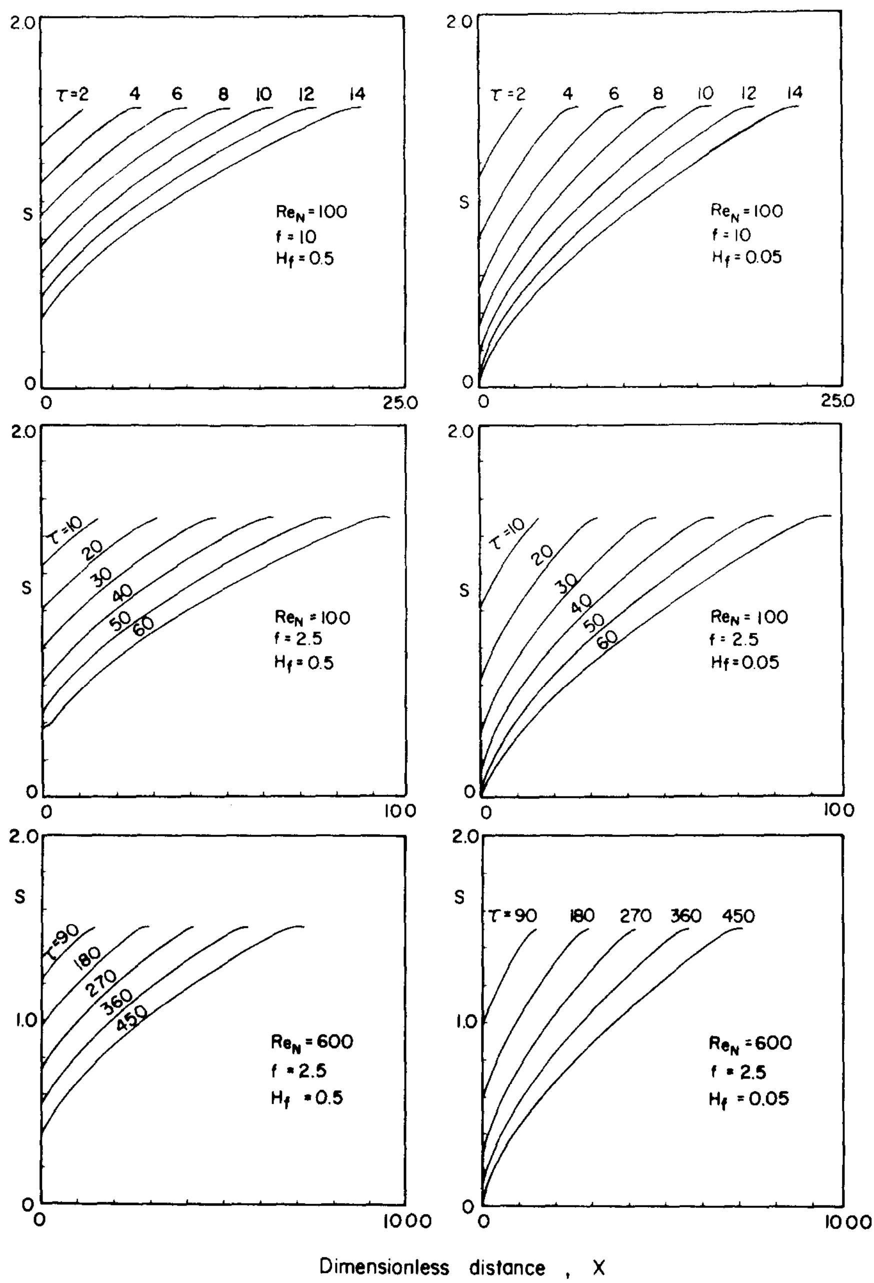

FIG. 4. Local and instantaneous variations of the interface velocity s for various operation conditions.

As indicated in Figs. 4–9, the distance (X_c) downstream, which is under the effect of the disturbance at $X = 0$, increases with the Reynolds number and decreases with the drainage rate $1/f$ (at $X = 0$). However, the enhancement in the transfer rates relative to the initial Nusselt values depend on the final condition of the film thickness at $X = 0$, whether $H_f \rightarrow 0.5$ or $H_f \rightarrow 0.05$. Clearly the latter yields a greater improvement.

Finally, it is to be noted that the heat-transfer coefficient is proportional to Q_w in the case of constant temperature at the wall (since the interface temperature, T_i, is constant at the saturation value). On the other hand, in the case of constant heat flux at the wall, the heat-transfer coefficient is proportional to $1/\Theta_w$.

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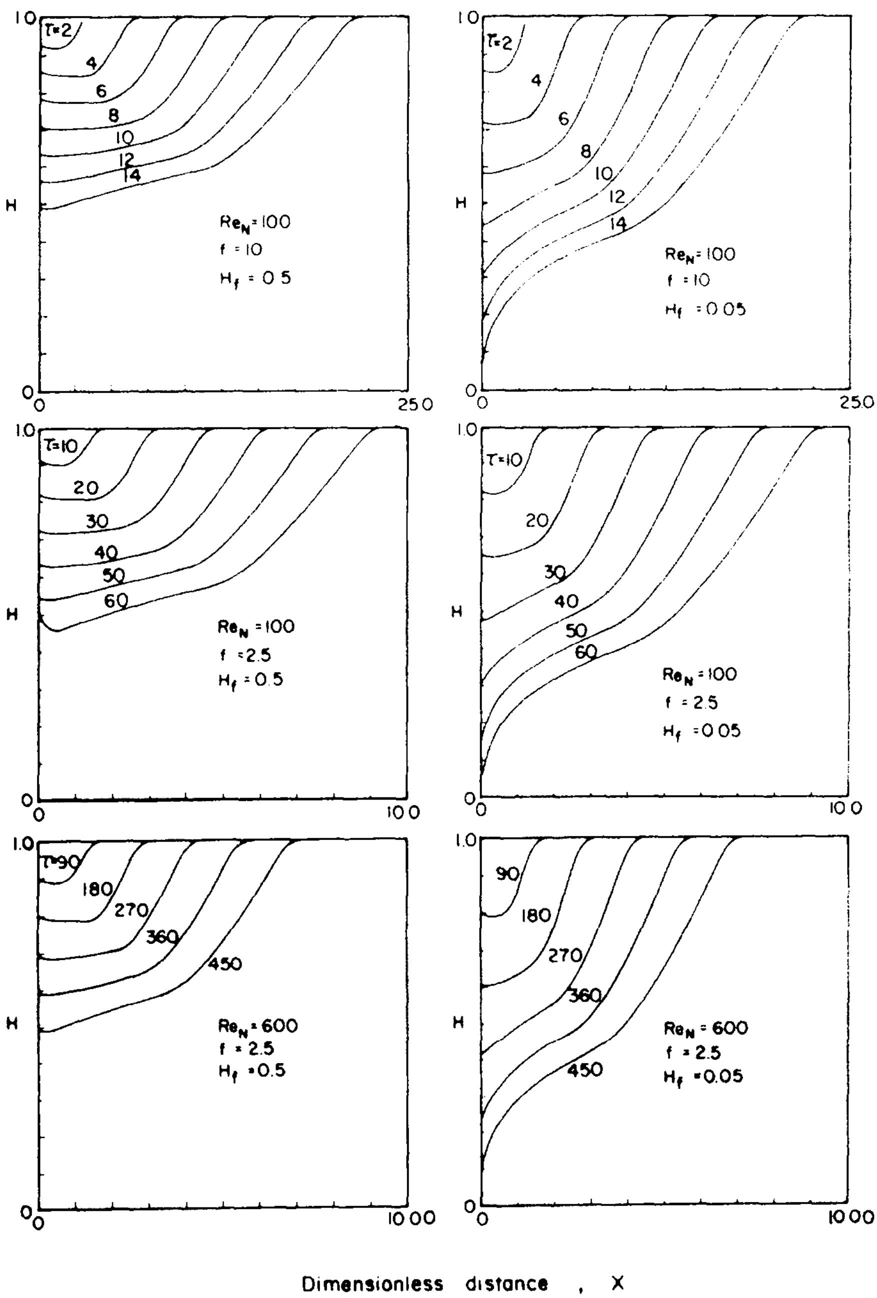

FIG. 5. Local and instantaneous variations of the film thickness, H for various operation conditions.

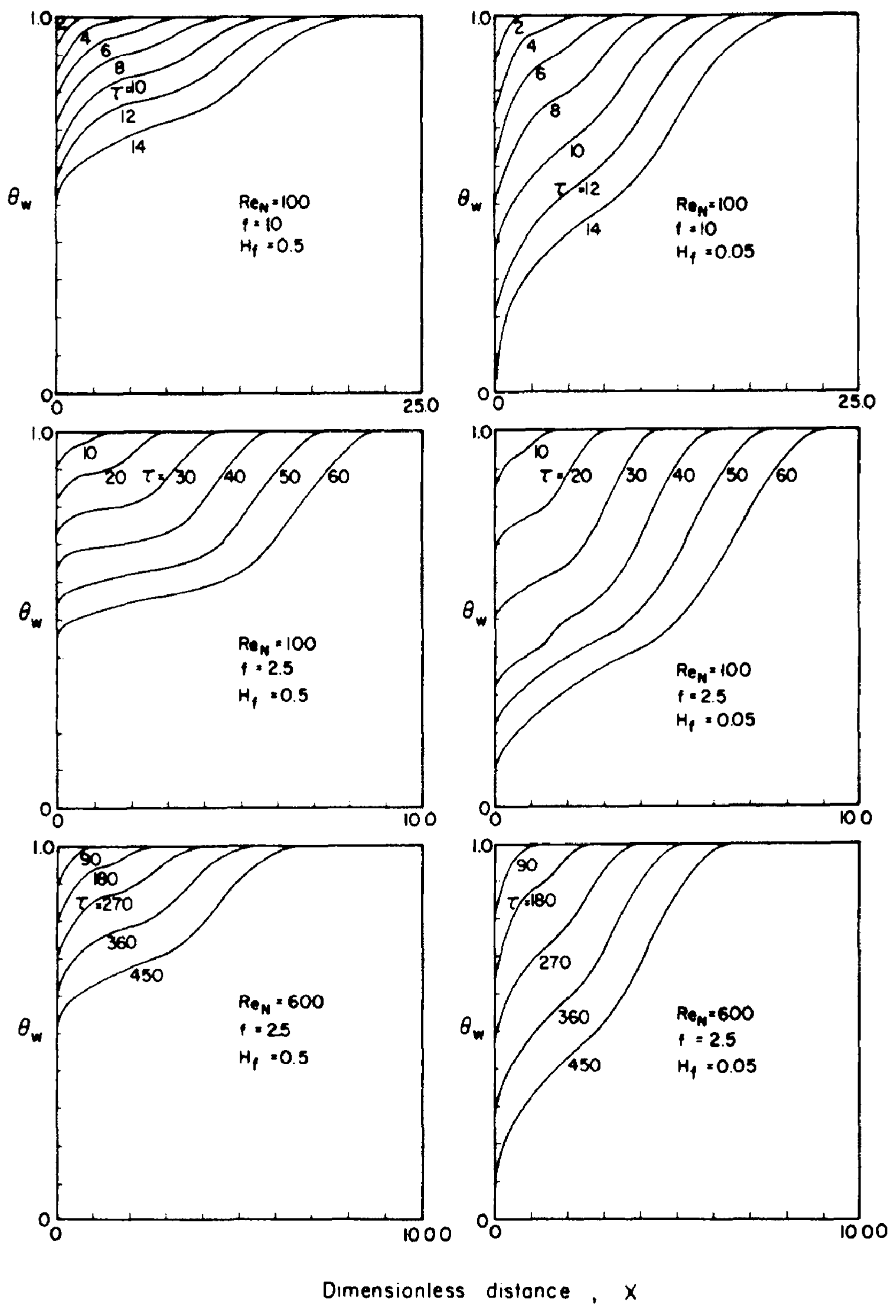

FIG. 6. Local and instantaneous variations of the temperature at the wall, T_w for various operation conditions.

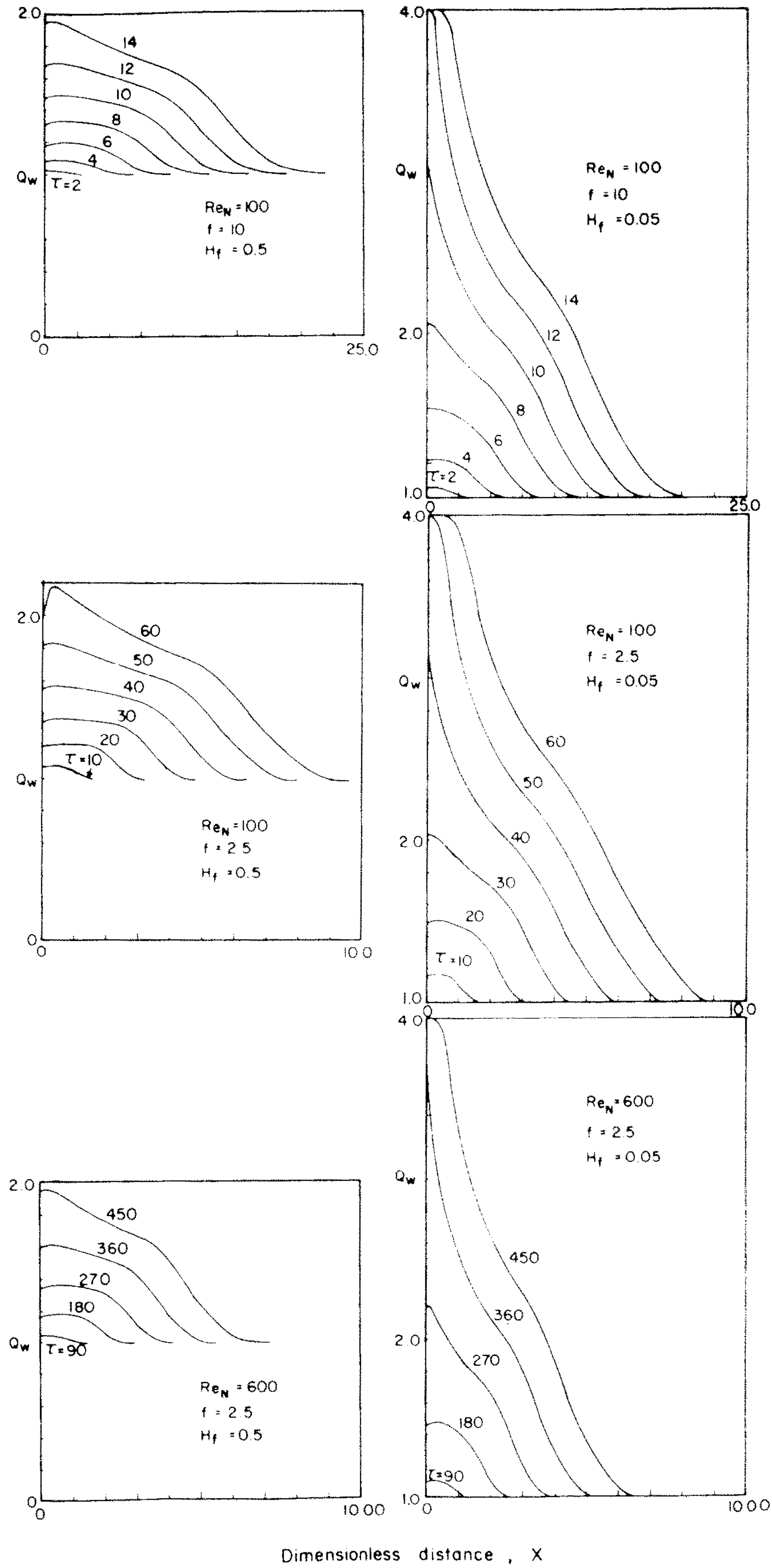

FIG. 7. Local and instantaneous variations of the heat flux at the wall, Q_w for various operation conditions.

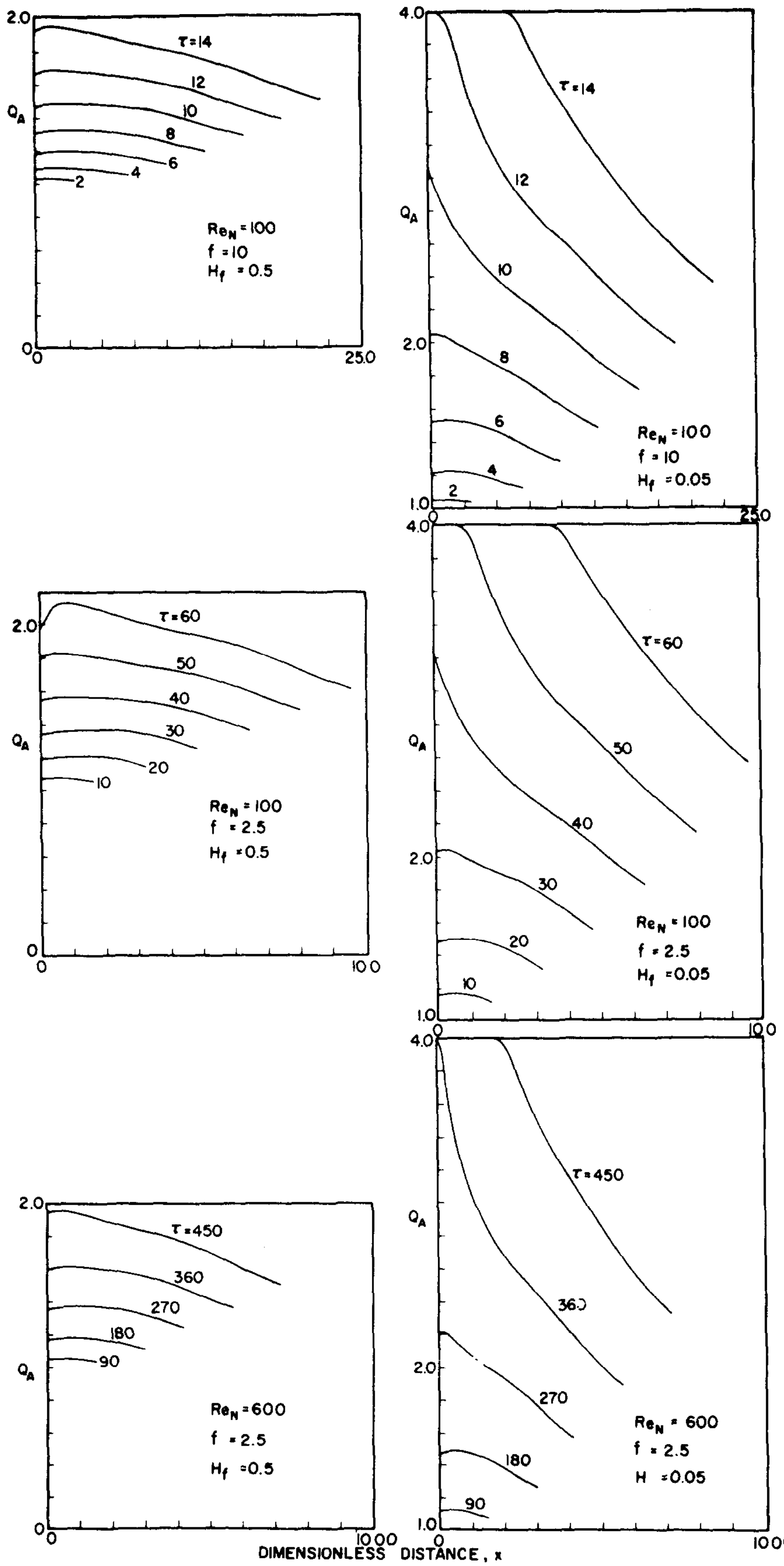

FIG. 8. Spatial average heat flux at different periods for various operation conditions.

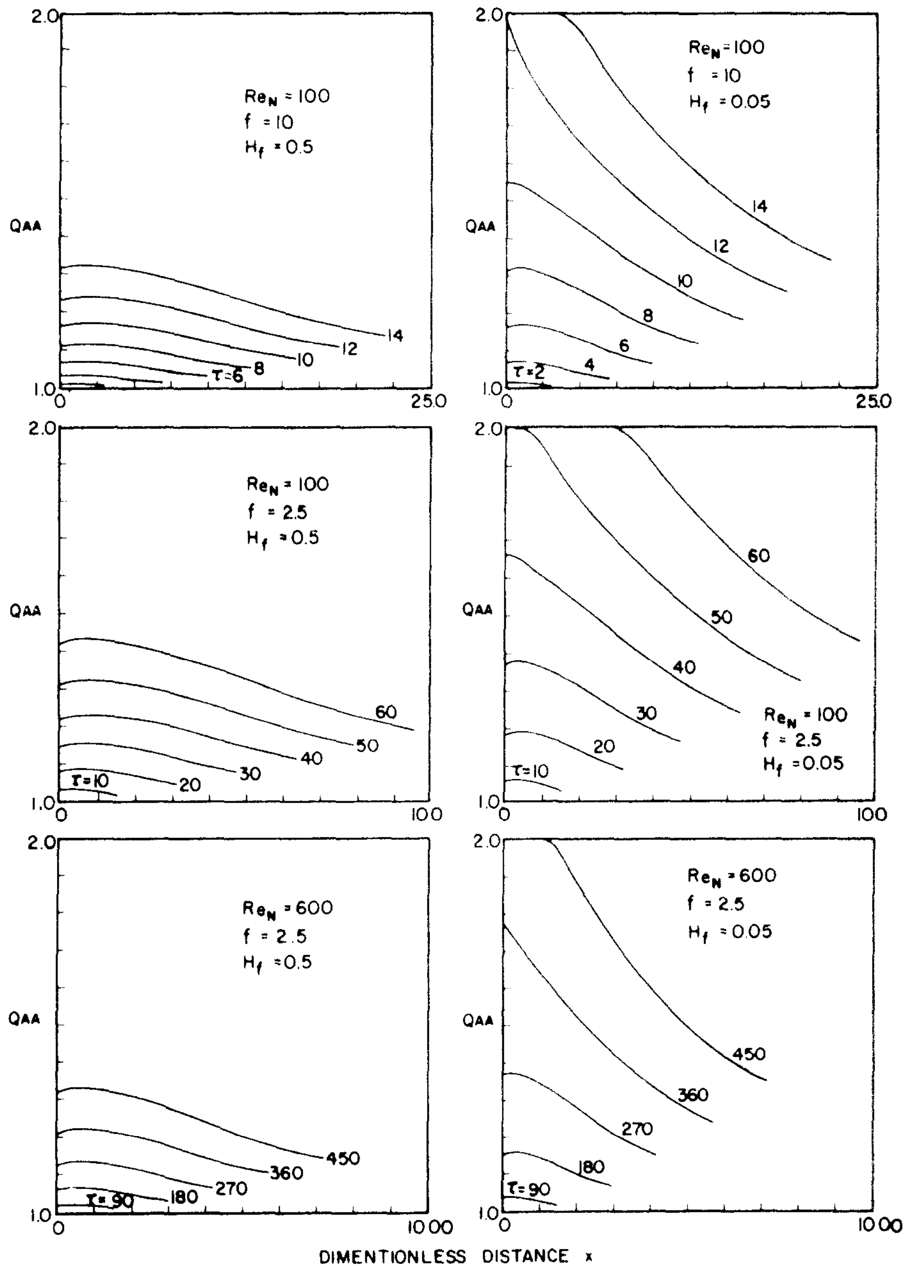

FIG. 9. Spatial and temporal average heat flux at different periods for various operation conditions.

REFERENCES

1. D. Moalem Maron and S. Sideman, Theoretical analysis of a horizontal condenser–evaporator tube, *Int. J. Heat Mass Transfer* **19**, 259 (1976).
2. D. Moalem Maron and S. Sideman, Theoretical analysis of a horizontal condenser–evaporator elliptical tube, *J. Heat Transfer* **97C**, 352 (1975).
3. S. Sideman, H. Horn and D. Moalem Maron, Transport characteristics of films flowing over horizontal smooth tubes, *Int. J. Heat Mass Transfer* **22**, 1165 (1979).
4. W. D. Campbell and J. C. Slattery, Flow in the entrance of a tube, *J. Bas. Engng* **85**, 41 (1963).
5. J. M. Dealy, A neglect effect in entrance flow analyses, *A.I.Ch.E.Jl* **11**, 745 (1965).
6. S. V. Patanker and D. B. Spalding, *Heat and Mass Transfer in Boundary Layers*, Morgan-Grampian, London (1962).
7. D. E. Rosenberg and J. D. Hellums, Flow development and heat transfer in variable viscosity fluids, *I/EC Fundamentals* **4**, 417 (1965).
8. H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York (1956).
9. E. M. Sparrow, S. H. Lin and T. S. Lundgren, Flow development in the hydrodynamic entrance region of tubes and ducts, *Physics Fluids* **7**, 338 (1964).
10. L. E. Scriven and R. L. Pigford, Absorption into an accelerating film, *A.I.Ch.E.Jl* **4**, 382 (1958).
11. H. L. Toor, Averaging of mass transfer rates, *A.I.Ch.E. Jl* **2**, 578 (1956).
12. N. A. Hassan, Laminar flow along a vertical wall, *J. Appl. Mech.* **34**, 535 (1967).

13. R. Haugen, Laminar flow along a vertical wall, *J. Appl. Mech.* **11**, 945 (1968).
14. D. F. Bruley, Predicting vertical film flow characteristics in the entrance region, *A.I.Ch.E. Jl* **11**, 945 (1965).
15. R. L. Cerro and S. Whitaker, Entrance region flows with a free surface: the falling liquid film, *Chem. Engng Sci.* **26**, 785 (1971).
16. N. S. Murty and V. V. H. K. Sastri, Acceleration laminar liquid film along an inclined wall, *Chem. Engng* **28**, 869 (1973).

CARACTERISTIQUES HYDRODYNAMIQUES ET DE TRANSFERT DANS DES FILMS LAMINAIRES TOMBANTS A INTERFACE LIBRE AVEC TRANSFERT THERMIQUE

Résumé—On présente une étude théorique des caractéristiques hydrodynamiques et de transport à la région initiale des films à surface libre. Les équations de continuité, du mouvement du liquide, et du transfert thermique à travers le film sont formulées de façon nouvelle en fonction des paramètres (usuellement mesurables) à l'interface en considérant celui-ci comme une ligne de collocation. Des termes d'inertie sont conservés comme dans les problèmes associés à des perturbations rapides. Une perturbation dépendante du temps est appliquée à l'entrée et les effets de sa propagation sur les caractéristiques du film et du transfert en aval sont évalués et discutés.

Ces formes collocatives des équations peuvent être intéressantes pour des analyses ultérieures des films.

HYDRODYNAMIK UND TRANSPORTEIGENSCHAFTEN BEI FILMEN MIT FREIER OBERFLÄCHE BEI ZEITABHÄNGIGEN STÖRUNGEN AM EINTRITT

Zusammenfassung—Es werden theoretische Berechnungen der Hydrodynamik und der Transporteigenschaften im Einlaufbereich von Filmen mit freier Oberfläche mitgeteilt. Die Bestimmungsgleichungen für Massenkontinuität, Flüssigkeitsbewegung und Wärmetransport durch den Film werden neu formuliert, und zwar mit den (gewöhnlich meßbaren) Parametern an der Grenzfläche als unabhängigen Variablen. Dabei wird letztere als sogenannte Kollokationslinie angesetzt. Trägheitsglieder wurden berücksichtigt, so wie es für Aufgabenstellungen mit schnellen Störungen gefordert wird. Es wird eine zeitabhängige Störung am Eintritt angesetzt und der Einfluß ihrer Fortpflanzung auf die Film- und Transporteigenschaften stromabwärts abgeschätzt und diskutiert. Die Kollokationsformen der Bestimmungsgleichungen können für weitere Filmberechnungen von Interesse sein.

ВЛИЯНИЕ ЗАВИСЯЩЕГО ОТ ВРЕМЕНИ ВОЗМУЩЕНИЯ ВО ВХОДНОЙ ОБЛАСТИ НА ГИДРОДИНАМИЧЕСКИЕ ХАРАКТЕРИСТИКИ И ПЕРЕНОСНЫЕ СВОЙСТВА СВОБОДНОЙ ПЛЕНКИ НА ПОВЕРХНОСТИ РАЗДЕЛА

Аннотация — Представлен теоретический анализ гидродинамических характеристик и переносных свойств на начальном участке пленок со свободной поверхностью. Основные уравнения сохранения массы, количества движения жидкости и переноса тепла поперек пленки выражены через (обычно измеряемые) параметры на границе раздела, причем последняя представлена в виде так называемой коллокационной линии. В уравнениях переноса сохранены инерционные члены, поскольку рассматривается задача с быстро меняющимися возмущениями. Использовано зависящее от времени возмущение на входе и проведена оценка его влияния на характеристики пленки и переносные свойства вниз по течению. По-видимому, основные уравнения в коллокационной форме могут представить интерес с точки зрения их использования при исследовании пленок.